Let $T \sim \text{unif}\{1, \dots, n\}$

$T \perp\!\!\!\perp (X^n, M, \hat{X}^n)$

$$\sum_{i=1}^{n} I(X_i ; \hat{X}_i)$$
$$= n I(X_T ; \hat{X}_T | T)$$
$$\geq n I(X_T ; \hat{X}_T)$$

$X_T \perp\!\!\!\perp T$ as X_i are i.i.d.

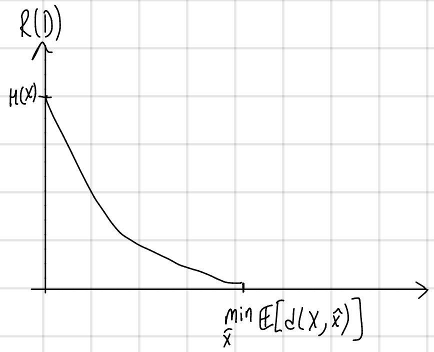
Lastly,

define $X = X_T$, $\hat{X} = \hat{X}_T$, notice $X \sim P_X$

Also "

$$\mathbb{E}[d(X, \hat{X})] = \mathbb{E}\left[\mathbb{E}_{T|(X,\hat{X})|X^n \hat{X}^n}[d(X, \hat{X})]\right]$$
$$= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} d(X^n, \hat{X}^n)\right]$$
$$\leq D$$

10/13/2016
Thursday



Let (R, D) be in the interior:

$\Longrightarrow \exists P_{\hat{X}|X}$ s.t.

$I(X;\hat{X}) < R$
$\mathbb{E}[d(X,\hat{X})] < D$

Codebook: Derive $P_{\hat{X}}(\hat{x}) = \sum_{x} P_X(x) P_{\hat{X}|X}(\hat{x}|x)$

$C = \{\hat{X}^n(m)\}_{m=1}^{2^{nR}}$ where $\hat{X}^n(m)$ are i.i.d. $\sim P_{\hat{X}}$ $\forall m$ and independent.

Encoder: Choose $\varepsilon > 0$, find any m s.t. $(X^n, \hat{X}(m)) \in \mathcal{T}_\varepsilon^{(n)}$ Transmit m
If error, chose any m.
(error occurs only if $\nexists m$!)

X^n: observed.

↳ complexity is in encoder (cf. Channel coding thm proof, where complexity is in decoder)

Decoder: Reconstruct $\hat{X}(m)$

Expected Distortion: $\mathbb{E}[d(X^n,\hat{X}^n)] = \underbrace{\mathbb{P}(\text{error})}_{\to 0} \underbrace{\mathbb{E}[d(X^n,\hat{X}^n)\mid \text{error}]}_{\leq \max_{x,\hat{x}} d(x,\hat{x}) \triangleq d_{max}} + \underbrace{(1-\mathbb{P}[\text{error}])}_{\leq 1} \underbrace{\mathbb{E}[d(X^n,\hat{X}^n)\mid \text{no error}]}_{\leq (1+\varepsilon)\underbrace{\mathbb{E}[d(X,\hat{X})]}_{<D} \ < D}$

if ε small enough.

$< D$ $\leftarrow$ (for n large and ε small enough)

$\mathbb{P}[\text{error}] \leq \mathbb{P}[X^n \notin T_\varepsilon^{(n)}] + \mathbb{P}[\mathcal{E} \mid X^n \in T_\varepsilon^{(n)}]$ $\qquad \mathcal{E}: \forall m,\ (X^n, \hat{X}^n(m)) \notin T_\varepsilon^{(n)}$

$\left(\mathbb{P}[(X^n,\hat{X}^n(1)) \notin T_\varepsilon^{(n)} \mid X^n \in T_\varepsilon^{(n)}]\right)^{2^{nR}} \leq \left(1-(1-\varepsilon)2^{-n(I(X;Y)+4\varepsilon)}\right)^{2^{nR}}$ (see prev. notes!) $\quad (1-x \leq e^{-x})$

$\leq e^{-(1-\varepsilon)2^{-n(I(X;Y)+4\varepsilon)}}$

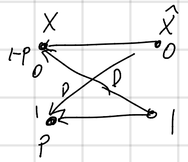
note:
correct way to do it

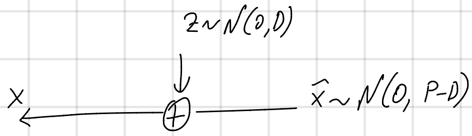
$\sum_{x^n} p_{X^n}(x^n)\, \mathbb{P}[(x^n,\hat{X}^n) \notin T_\varepsilon^{(n)}]$

$= \sum_{x^n \notin T_\varepsilon^{(n)}}$

$\sum_{x^n \in T_\varepsilon^{(n)}} p_{X^n}(x^n)\, \underbrace{\mathbb{P}(\forall m, (x^n, \hat{X}^n(m)) \notin T_\varepsilon^{(n)})}_{\leq e^{-(1-\varepsilon)2^{n(R-I(X;\hat{X})-4\varepsilon)}} \to 0}$ $\qquad$ Let $\varepsilon < \frac{R - I(X;\hat{X})}{4}$

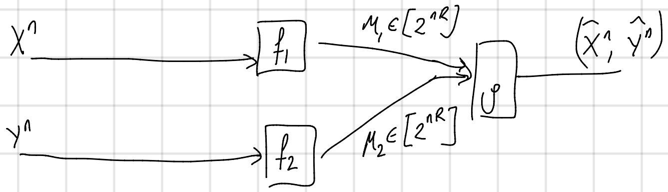
Binary - Hamming $\qquad R(D) = h(p) - h(D) \quad$ for $D \leq p \leq \frac{1}{2}$

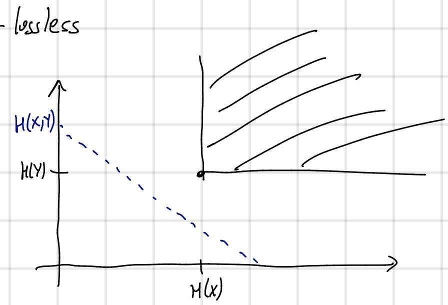




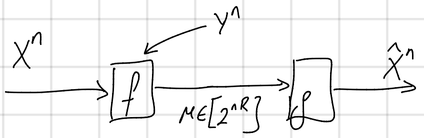
Slepian-Wolf (1973)

Distributed Source Coding





(X^n, Y^n) are i.i.d. pairs $\quad P_{X^nY^n} = \prod_{i=1} P_{XY}$



Sequence $\quad Y=0 \quad X_i \sim P_{X|Y=0}$

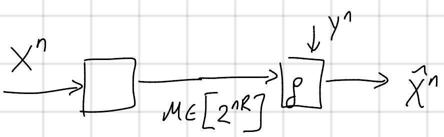
$H(X|Y=0)$

inf of achievable rates $= H(X|Y)$

Achievability: Enumerate conditionally typical set, or interleaving, or Random Encode (Hash function)

Converse: $X^n - (Y^n, M) - \hat{X}^n$, Assume R is achievable $\Rightarrow \mathbb{P}[X^n \neq \hat{X}^n] < \varepsilon$

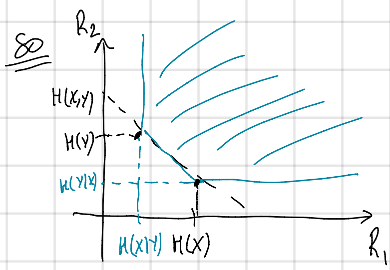
$$
\begin{aligned}
nR &\geq H(M) \\
&\geq H(M|Y^n) \\
&\geq I(X^n; \hat{X}^n) \quad \text{D.P.I.} \\
&= \underbrace{H(X^n|Y^n)}_{nH(X|Y)} - \underbrace{H(X^n|\hat{X}^n, Y^n)}_{\text{Fano}}
\end{aligned}
$$



still $R \geq H(X|Y)$

Thm: Closure of Achievable (R_1, R_2) rate pairs is

$$
(R_1, R_2): \quad
\begin{aligned}
R_1 &\geq H(X|Y) \\
R_2 &\geq H(Y|X) \\
R_1 + R_2 &\geq H(X,Y)
\end{aligned}
$$



"Random Binning" (Cover 1975)

Codebook: For each $x^n \in \mathcal{X}^n$, randomly (uniform) independently assign $M_1(x^n) \in \{1, 2, \dots, 2^{nR_1}\}$

for each $y^n \in \mathcal{Y}^n$, randomly (uniform) independently assign $M_2(y^n) \in \{1, 2, \dots, 2^{nR_2}\}$

X and Y codebooks are independent.

Encoder: Transmit $M_1(X^n)$ and $M_2(Y^n)$

Decoder: (let ε be small): Find unique (x^n, y^n) such that
$$(x^n, y^n) \in \mathcal{T}_\varepsilon^{(n)}$$
$$M_1(x^n) = M_1$$
$$M_2(y^n) = M_2$$

Error events: 1. $(X^n, Y^n) \notin \mathcal{T}_\varepsilon^{(n)}$ A.E.P.

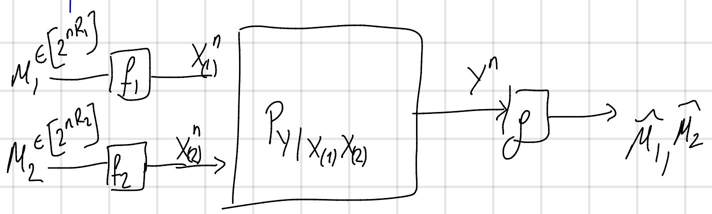
2. $\exists (x^n, y^n) \neq (X^n, Y^n)$ s.t. 3 requirements hold.

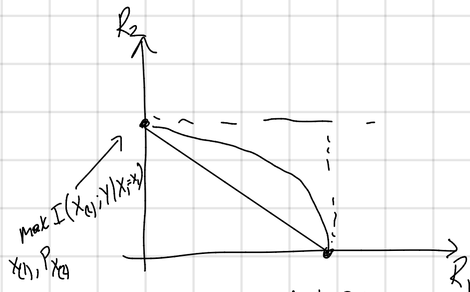
Union bound

$$\mathbb{P}(\text{error}_2) \leq \sum_{\substack{(x^n, y^n) \in \mathcal{T}_\varepsilon^{(n)} \\ (x^n, y^n) \neq (X^n, Y^n)}} \mathbb{P}[M_1(x^n) = M_1]\, \mathbb{P}[M_2(y^n) = M_2]$$

$$\leq \underbrace{\sum_{\substack{(x^n, y^n) \in \mathcal{T}_\varepsilon^{(n)} \\ x^n \neq X^n,\, y^n \neq Y^n}}}_{|\mathcal{T}_\varepsilon^{(n)}(X, Y)|} 2^{-n(R_1 + R_2)} + \underbrace{\sum_{\substack{x^n, y^n \in \mathcal{T}_\varepsilon^{(n)} \\ x^n = X^n,\, y^n \neq Y^n}}}_{\mathcal{T}_\varepsilon^{(n)}(Y | X^n)} 2^{-nR_2} + \sum_{\substack{x^n, y^n \in \mathcal{T}_\varepsilon^{(n)} \\ x^n \neq X^n,\, y^n = Y^n}} 2^{-nR_1}$$

Multiple Access Channel





$$C_1 = \max_{x_{(2)}, P_{X_{(1)}}} I(X_{(1)}; Y | X_{(2)} = x_2)$$